LINE-TRANSITIVE COLLINEATION GROUPS OF FINITE PROJECTIVE SPACES

BY

WILLIAM M. KANTOR[†]

ABSTRACT

A collineation group Γ of $PG(d, q)$, $d \geqq 3$, which is transitive on lines is shown to be 2-transitive on points unless $d = 4, q = 2$ and $|\Gamma| = 31 \cdot 5$.

The purpose of this paper is to prove the following result.

THEOREM 1. *Let $\Gamma \leqq P\Gamma L(d+1,q)$ be transitive on the lines of the projective space $PG(d,q)$, where $d \geqq 3$. Then either Γ is 2-transitive on the points of the space or $d = 4$, $q = 2$ and $|\Gamma| = 31 \cdot 5$.*

Thus, line-transitive collineation groups are generally 2-transitive. The determination of all 2-transitive collineation groups of finite projective spaces is a difficult question (see [14], [17]); our proof gives no information about them.

This theorem was motivated by some recent results of D. Perin [11]. He needed it in order to complete his results in characteristic 2. For completeness, we will state Perin's result after Theorem 1 is plugged in:

THEOREM 2. *Let $\Gamma \leqq P\Gamma L(d+1,q)$ be transitive on the planes of $PG(d,q)$, where $d \geqq 4$. Then $\Gamma \geqq PSL(d+1,q)$, except perhaps when $q = 2$ and d is odd.*

Perin also obtained analogous results for transitivity on higher dimensional subspaces— and no ambiguity then occurs when $q = 2$ —but Theorem 1 is not needed except for plane-transitivity. Our proof will be completely different from Perin's. He concentrated on primitive divisors of $q^{d-1} - 1$, whereas we are mostly concerned with 2-groups.

[†] Research supported in part by NSF Grant GP 28420.

Received October 26, 1971

PROOF OF THEOREM 1. We will assume that Γ is not 2-transitive on points. It follows from line-transitivity that, for each line L, Γ_L is not 2-transitive on L; this fact will be used frequently throughout the proof.

The points and lines form a design with $v=(q^{d+1}-1)/(q-1)$, $r=(q^d-1)/(q-1)$, $k=q+1$, $\lambda=1$ and $b=rv/k$. Let p be the prime dividing q.

Set $\Delta=\Gamma\cap PGL(d+1,q)$. Clearly $\Delta\neq 1$.

NOTATION. If X is a subspace, dim X is its dimension (points have dimension 0) and $\Delta(X)$ is its pointwise stabilizer. If Σ is a subset of Γ, $F(\Sigma)$ is its set of fixed points and $N(\Sigma)^{F(\Sigma)}$ the permutation group induced by its normalizer on $F(\Sigma)$.

We proceed in a series of steps.

I) Γ *is primitive on points.*

PROOF. (Compare Higman and McLaughlin [5]; see also [6] and [2, p. 79].) Let C be an imprimitivity class of size c, $1<c<v$, and set $n=v/c$. Let $x\in C$.

Since Γ is transitive on v points and b lines, all line-orbits of Γ_x have lengths divisible by $b/(b,v)=r/e$, where $e=k/(v,k)$. Let Γ_x have s orbits of lines on x of lengths $w_i r/e$, $i=1,\cdots,s$. Then $t_i=|L_i\cap C|$ is independent of the choice of L_i in the ith orbit. Thus,

$$c-1=\Sigma_i(t_i-1)w_i r/e=\frac{r}{e}(t-1),$$

where $t-1=\Sigma_i(t_i-1)w_i$. Clearly $t\geqq 2$. Now

$$v-n=\frac{r}{e}n(t-1)$$

$$v-1=\frac{r}{e}e(k-1)$$

$$n-1=\frac{r}{e}[e(k-1)-n(t-1)].$$

Consequently, $e(k-1)>n(t-1)>n-1\geqq r/e$. Since $e\leqq q+1$ we have $(q+1)^2q>(q^d-1)/(q-1)$. Then $d=3$ or 4. If $d=3$ then $e=1$ and $k-1>r$, which is not the case. Now $d=4$ and $e=k$. Here $e(k-1)>2r/e$ is impossible. so we must have $t=2$, $n-1=r/e$, and $e(k-1)-n(t-1)=1$. Then $n=1+r/e=1+(q^2+1)$ and $n=e(k-1)-1=(q+1)q-1$.

Consequently, $q=3$, $n=11$, and $c=11$. Choose j with $t_j>1$. As $1=t-1=\Sigma_i(t_i-1)w_i$, we must have $t_j=2$ and $w_j=1$. Since Γ_x acts on the $10=w_j r/e$ points of $C-\{x\}$, it follows that the group Γ_C^C induced by Γ on C is 2-transitive of degree 11. A Sylow 11-subgroup Σ^* of Γ is cyclic and has a subgroup Σ of order 11 fixing each class C. Thus Σ is in the stabilizer Π in Γ of all 11 clases. Since $N_{P\Gamma L(5,3)}(\Sigma^*) = N_{P\Gamma L(5,3)}(\Sigma)$ is Frobenius of order $11^2.5$, $N_\pi(\Sigma) = C_\pi(\Sigma)$, so Π has a normal 11-complement. Then $\Sigma^C = \Pi^C \lhd \Gamma_C^C$ which is absurd.

II) *If* $|\Delta|$ *is odd then* $d = 4$, $q = 2$ *and* $|\Gamma| = 31 \cdot 5$.

Proof. By (I) and the Feit-Thompson Theorem [3], Γ has a normal elementary abelian l-subgroup $\Lambda \leqq \Delta$ transitive on points (where $l > 2$ is a prime). The complete inverse image of Λ in GL$(d+1,q)$ acts fixed-point-freely, so that Λ is cyclic and $v = |\Lambda| = l$. Then $b = l(q^d - 1)/(q^2 - 1)$ divides $|N_{P\Gamma L(d+1,q)}(\Lambda)| = l(d+1)i$, where $q = p^i$. Now (II) follows easily. (Compare Lüneburg [9] for this part of the proof of Theorem 1.)

From now on we will assume that $|\Delta|$ is even.

III) $p > 2$.

Proof. Suppose that $p = 2$ and let $\sigma \in \Delta$ be an involution. Then $F(\sigma)$ is a subspace. Suppose first that $\dim F(\sigma) \leqq 1$. Then $\dim F(\sigma) = 1$ and $d = 3$. Each plane $E \supset F(\sigma)$ is fixed by σ. Each line in E determines a conjugate of σ fixing that line pointwise. Thus, the global stabilizer Γ_E of E induces a collineation group Π of E such that each line of E is fixed pointwise by an involution in Π fixing just the points of that line. By [2, p. 193], there is a line L of E such that $\Pi(L)$ is transitive on $E - L$ and Π_L is transitive on L. It follows that, for any line $M \neq L$ of E, $\Pi(L)_M$ is transitive on $M - L \cap M$. Thus, Γ_M is 2-transitive on M, so Γ is 2-transitive on points, which is not the case.

We can thus find a subspace X with $\dim X \geqq 2$ and $|\Delta(X)|$ even. Choose X such that $\dim X \geqq 2$, $X = F(\Sigma)$ for some 2-group $\Sigma \neq 1$, and $\dim X$ is minimal for such a subspace. We may assume that Σ is Sylow in $\Delta(X)$.

Consider a line $L \subset X$. Clearly $|\Delta_L^L|$ is even. Let $\Delta_L \geqq \Lambda \rhd \Sigma$ with $|\Lambda : \Sigma| = 2$. Then Λ^X is an involution. By our choice of X and Σ, $\dim F(\Lambda) \leqq 1$. Since $\dim X \geqq 2$, $\dim F(\Lambda) = 1$. Then Σ is not Sylow in $\Delta(L)$, so we can choose Λ with $L = F(\Lambda)$. Clearly Λ^X fixes each plane E of X containing L. Since L can be taken as any line of E, by considering $N(\Sigma)_E^E$ we obtain the same contradiction as in the first paragraph.

IV) Δ *contains an involution which has fixed points.*

Proof. Suppose not. Let $\hat{\Delta}$ and $\hat{\Gamma}$ be the complete inverse images of Δ and Γ in $\Gamma L(d+1,q)$, so that $\hat{\Delta}$ is a group of linear transformations of a $d+1$-dimensional GF(q)-space V. Our hypothesis is that $\hat{\Delta}$ has just one involution, and hence has cyclic or generalized quaternion Sylow 2-subgroups.

Clearly v is even. By (I), $O(\Delta) = 1$ and Δ is not a 2-group. By Burnside's transfer theorem, $\hat{\Delta}$ has generalized quaternion Sylow 2-subgroups. The Gorenstein-Walter Theorem [4] thus implies that either (i) $\Delta \approx A_7$ or (ii) $\Delta \rhd \Delta^*$ with $\Delta^* \approx PSL(2,m)$ for some odd m. By (I), $C_\Gamma(\Delta^*) = 1$, so either (i) $\Gamma \approx A_7$ or S_7, or (ii) Γ is isomorphic to a subgroup of $P\Gamma L(2,m)$ containing PSL$(2,m)$.

Suppose first that $p \,\big|\, |\Gamma|$ (where p is again the prime dividing q). Since $p \nmid b$, each normal subgroup of Γ of index a power of p must be transitive on lines. Consequently, we may assume that Δ contains a Sylow p-subgroup Π of Γ. Clearly Π fixes a point x. Note that $p \,\big|\, |\Gamma_M^M|$ for any line M; for if $1 \neq \pi \in \Pi$. then π fixes some subspace X containing $F(\pi)$ as a hyperplane, and π induces a nontrivial elation on X. We now claim that x is the only fixed point of Π; for if Π fixes $y \neq x$ and $L = xy$, then $\Pi \leqq \Delta(L)$, which is impossible since $p \,\big|\, |\Gamma_L^L|$ and Π is Sylow in Γ. In particular, $N(\Pi)$ fixes x, so $N_\Delta(\Pi)$ has odd order (ii) must hold, as otherwise $|S_7| \geqq (7^4 \cdot 1)(7^3 - 1)/(7^2 - 1)(7 - 1) > |S_7|$. As above we may assume $\Pi \leqq \Delta^*$. An examination of the Sylow subgroups of Δ^* shows that $|N_{\Delta^*}(\Pi)|$ can be odd only if $p \mid m$ and $m \equiv 3 \pmod 4$. There is a fixed line L of Π, and $\Pi(L)$ is Sylow in $\Delta(L)$. By the Frattini argument, $\Delta_L = \Delta(L) N_{\Delta_L}(\Pi(L))$. Since Δ_L^L has even order (by (II)), so does $N_\Delta(\Pi(L))$. But $m \equiv 3 \pmod 4$, so $\Pi(L)$ must be trivial. Consequently, $m = |\Pi| \leqq |L| - 1 = q$. Now

$$\frac{q^{d+1}-1}{q-1} \leqq |\Delta^*| < m^3 \leqq q^3,$$

contradicting the fact that $d \geqq 3$.

Thus, $p \nmid |\Gamma|$. In particular, since $3 \,\big|\, |\Delta|$, $p \neq 3$ and $q \geqq 5$. Note that d is odd as $v = (q^{d+1} - 1)/(q - 1)$ is even. Also

$$\text{1)} \qquad |\Gamma| \geqq b = \frac{(q^{d+1}-1)(q^d-1)}{(q-1)(q^2-1)} > q^{\frac{1}{2}(d+1)} q^{d-1} \geqq q^4.$$

If (i) holds, $p \neq 3, 5, 7$. By (1), $7\,! > q^4 \geqq 11^4$, which is false.

Thus, (ii) holds and

$$\text{2)} \qquad m^4 > |\Gamma| > q^{\frac{1}{2}(3d-1)} \geqq q^4.$$

If $m < 9$ then, by (2), $7^3 \geqq |\Gamma| > 5^4$, which is false. Similarly, if $m = 9$ then p

must be 7 (as $p \nmid |\Gamma|$) and $2 \cdot 9^3 > |\Gamma| > 7^4$, while if $m = 11$ then $11^3 > |\Gamma| > 7^4$. Thus, $m \geqq 13$.

By [13], $\hat{\Delta}$ has a normal subgroup $\tilde{\Delta} \approx \mathrm{SL}(2,m)$. Let K be the algebraic closure of $\mathrm{GF}(q)$. Then $V \otimes K$ is a $d+1$-dimensional $\tilde{\Delta}$-module. Let W be any nontrivial irreducible constituent of $V \otimes K$. Then $d + 1 \geqq \dim_K W = e$. On the other hand, W is an absolutely irreducible $\tilde{\Delta}$-module of characteristic p, where $p \nmid |\tilde{\Delta}|$. Consequently, W can be lifted to a complex irreducible $\tilde{\Delta}$-module of dimension e.

By [13], each nontrivial complex irreducible representation of $\mathrm{SL}(2,m)$ has degree $\geqq (m-1)/2$. Thus, $d+1 \geqq e \geqq (m-1)/2$.

Now (2) yields $m^{16} > q^{2(3d-1)} \geqq 5^{3m-11}$. However, this is false for $m = 13$, and for $x \geqq 13$ the function $16 \log x - (3x - 11)\log 5$ is decreasing. This contradiction proves (IV).

V) *The following conditions hold*:

a) *Each line L determines a unique point w_L of L such that Γ_L fixes w_L and Δ_L is transitive on $L - \{w_L\}$, and*

b) $|\Delta(L)| \equiv 0 \pmod p$.

PROOF. By (IV) we can find an involution $\sigma \in \Delta$ with fixed points. Then $F(\sigma) = Y_1 \cup Y_2$ with Y_1, Y_2 disjoint subspaces spanning the whole space. Suppose first that both of these have dimension $\leqq 1$. Then both have dimension 1 and σ fixes all planes $E \supset Y_1$. For each line L of E there is a conjugate of σ fixing just the points of L. By [2, 196] Δ_E^E is (c, M)-transitive for some $c \in M \subset E$, so (b) holds. If $c \in L \subset E$ and $L \neq M$ then Δ_{cL} is transitive on $L - \{c\}$. Since Γ is not 2-transitive, Γ_L must fix c, so (a) holds.

Now suppose $\dim Y_1 \geqq 2$. Let $\Sigma \neq 1$ be a 2-group in Δ maximal with respect to fixing some plane pointwise; let X be a subspace of dimension $\geqq 2$ fixed pointwise by Σ and not properly contained in any other such supspace. Then Σ is Sylow in $\Delta(X)$.

Suppose $\Delta \geqq \Lambda \rhd \Sigma$, where $|\Lambda : \Sigma| = 2$ and Λ fixes some point $x \in X$. Let $\lambda \in \Lambda - \Sigma$. Then Σ fixes X and X^λ pointwise, while $x \in X \cap X^\gamma$. The choice of X then forces $X = X^\lambda$, so Λ fixes X. Also, the choice of Σ shows that Λ fixes no plane of X pointwise.

Consider a line L of X. There is a conjugate of σ fixing just 2 points x, y of L. We can thus find Λ with $\Delta_{xy} \geqq \Lambda \rhd \Sigma$ and $|\Lambda : \Sigma| = 2$. Then Λ^X is an involution having fixed points. Let $F(\Lambda) \cap X = X_1 \cup X_2$ with X_1, X_2 subspaces.

By our choice of Σ, both X_1 and X_2 have dimension $\leqq 1$, and hence at least one of them has dimension 1. Then Σ has smaller order than a Sylow 2-subgroup of $\Delta(L)$, so we can choose our Λ so that $F(\Lambda) \cup X = L \cap X_0$ for some suspbace X_0. All planes E of X containing L are fixed by Λ^X. Since L can be taken to be any line of E, (a) and (b) hold as in the first paragraph.

We now complete the proof of the theorem by playing the same game with p-groups as we have been playing with 2-groups. We may assume that d is chosen as small as possible in order to obtain a contradiction.

Let Π be a p-subgroup of Δ maximal with respect to fixing at least 2 points. By (Vb), $\Pi \neq 1$. Also, $F = F(\Pi)$ is a subspace of dimension $\geqq 1$. By [8, pp. 400–401], $N_\Delta(\Pi)$ is transitive on F. Thus, $\dim F \geqq 2$ by (Va).

Let L be any line of F. Let $\Phi \geqq \Pi$ by a Sylow p-subgroup of Δ_L. Clearly $\Phi \triangleright \Pi$. By (Va), $|\Phi/\Pi| = q$. Φ acts on F, and by our choice of Π each element $\phi \neq 1$ of Φ^F fixes just one point of F. Here ϕ fixes L and w_L. If ϕ fixes a line $L' \neq L$ of F, it fixes a point of L', so that $w_L \in L'$, L and L' span a plane, and ϕ fixes more than one line and hence more than one point of this plane, which is not the case. Thus, each line of F is fixed by a p-element of Δ fixing no other line of F. By Gleason's Lemma, $N_\Delta(\Pi)$ is transitive on the lines of F. Clearly, $N_\Delta(\Pi)^F$ is not 2-transitive. The minimality of d then implies that $\dim F = 2$.

Now $N_\Delta(\Pi)^F$ is a transitive subgroup of PGL$(3,q)$ which is not 2-transitive. By [10] or [1], $N_\Delta(\Pi)^F$ contains a normal cyclic subgroup and, if $x = w_L$, $|N_\Delta(\Pi)^F_x| = 1$ or 3. However, Φ^F fixes x and has order q. Thus, $q = 3$. It follows that $\Gamma = \Delta$ and $N(\Pi)^F_x$ has precisely $(13-1)/3 = 4$ orbits on $F - \{x\}$.

We now show that $N(\Pi)_x$ has at most 3 orbits on $F - \{x\}$. To see this, note that the number of point-orbits $\neq \{x\}$ of Γ_x is the number of orbits of Γ of ordered pairs of distinct points. By line-transitivity, if L is a line, the latter number is just the number of orbits of Γ_L of ordered pairs of distinct points of L. By (Va), Γ_L is transitive on $L - \{w_L\}$, and by (IV), Γ_L is even 2-transitive on the 3 points of $L - \{w_L\}$. Thus, Γ_L has precisely 3 orbits of ordered pairs of distinct points of L.

Let $\{x\}$, $A_1(x)$, $A_2(x)$, $A_3(x)$ be the point-orbits of Γ_x. If $y \in F \cap A_i(x)$ for some i, then Π is a Sylow p-subgroup of Γ_{xy}. It follows that $N(\Pi)_x$ is transitive on $F \cap A_i(x)$. Consequently, $N(\Pi)_x$ has at most 3 orbits on $F - \{x\}$, which is ridiculous.

This contradiction completes the proof of the theorem.

REFERENCES

1. D. M. Bloom, *The subgroups of PSL (3, q) for odd q*, Trans. Amer. Math. Soc. **127** (1967), 150–178.

2. P. Dembowski, *Finite geometries*, Springer, Berlin-Heidelberg-New York, 1968.

3. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 771–1029.

4. D. Gorenstein and J. H. Walter, *The characterization of finite groups with dihedral Sylow 2-subgroups*, I, II, III, J. Algebra **3** (1965), 85–151, 218–270, 354–393.

5. D. G. Higman and J. E. McLaughlin, *Geometric ABA-groups*, Illinois J. Math. **5** (1961), 382–397.

6. W. M. Kantor, *Automorphism groups of designs*, Math. Z. **109** (1969), 246–252.

7. W. M. Kantor, *On 2-transitive collineation groups of finite projective spaces* (to appear in Pacific J. Math.).

8. D. Livingstone and A. Wagner, *Transitivity of finite permutation groups on unordered sets*, Math. Z. **90** (1965), 393 403.

9. H. Lüneburg, *Fahnenhomogene Quadrupelsysteme*, Math. Z. **89** (1965), 82–90.

10. H. H. Mitchell, *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc. **14** (1911), 207–242.

11. D. Perin, *On collineation groups of finite projective spaces*, Math. Z. **126** (1972), 135–142.

12. F. C. Piper, *Elations of finite projective planes*, Math. Z. **82** (1963), 247–258.

13. I. Schur, *Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene lineare substitutionen*, J. Reine Angew. Math. **132** (1907), 85 137.

14. A. Wagner, *On collineation groups of finite projective spaces* I, Math. Z. **76** (1961), 411–426.

UNIVERSITY OF OREGON
EUGENE, OREGON, U.S.A.
AND
UNIVERSITY OF ILLINOIS
CHICAGO, ILLINOIS, U.S.A.